

CHARACTERISTIC MATRICES OF CELLULAR AUTOMATA WITH RULE 60 AND INTERMEDIATE BOUNDARY CONDITION

JAE-GYEOM KIM*

ABSTRACT. We investigate periodicities of characteristic matrices of cellular automata configured with rule 60 and intermediate boundary condition.

1. Introduction

Cellular automata have been demonstrated by many researchers to be a good computational model for physical systems simulation since the concept of cellular automata first introduced by John Von Neumann in the 1950's. And some researchers have studied on cellular automata with intermediate boundary condition [1,2,4,5,7].

In this note, we will investigate periodicities of characteristic matrices of cellular automata configured with rule 60 and intermediate boundary condition.

2. Preliminaries

A cellular automaton (CA) is an array of sites (cells) where each site is in any one of the permissible states. At each discrete time step (clock cycle) the evolution of a site value depends on some rule (the combinational logic) which is a function of the present states of its k neighbors for a k -neighborhood CA. For a 2-state 3-neighborhood CA, the evolution of the (i) th cell can be represented as a function of the present states of $(i - 1)$ th, (i) th, and $(i + 1)$ th cells as: $x_i(t + 1) = f\{x_{i-1}(t), x_i(t), x_{i+1}(t)\}$, where f represents the combinational logic. For such a CA, the modulo-2 logic is always applied.

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For a 2-state 3-neighborhood CA there are 2^3 distinct neighborhood configurations and 2^{2^3} distinct mappings from all these neighborhood configurations to the next states, each mapping representing a CA rule. The CA, characterized by a rule known as rule 60, specifies an evolution from the neighborhood configurations to the next states as;

$$\begin{array}{cccccccc} 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{array} .$$

The rule name 60 comes from that 00111100 in binary system is 60 in decimal system. The corresponding combinational logic of rule 60 is

$$x_i(t+1) = x_{i-1}(t) \oplus x_i(t),$$

that is, the next state of (i)th cell depends on the present states of its left and self neighbors.

If in a CA the same rule applies to all cells, then the CA is called a uniform CA; otherwise the CA is called a hybrid CA. There can be various boundary conditions; namely, null (where extreme cells are connected to logic '0'), intermediate (where the 2nd right cell of the leftmost cell of a 3-neighborhood CA is assumed to be the left neighbor of the leftmost cell of the CA and the 2nd left cell of the rightmost cell of the CA is assumed to be the right neighbor of the rightmost cell of the CA), periodic (where extreme cells are adjacent), etc. If in a CA the neighborhood dependence is only XOR logic, then it is called a noncomplemented CA. And the number of cells of a CA is called the length of a CA.

The characteristic matrix T of a CA is the transition matrix of the CA. The next state $f_{t+1}(x)$ of a linear CA is given by $f_{t+1}(x) = T \times f_t(x)$, where $f_t(x)$ is the current state and t is the time step. If all the states of the CA form a single or multiple cycles, then it is referred to as a group CA.

LEMMA 2.1 ([3]). *A noncomplemented CA is a group CA if and only if $T^m = I$ where T is the characteristic matrix of the CA, I is the identity matrix and m is a positive integer.*

LEMMA 2.2 ([6]). *Let H be a uniform CA of length n configured with rule 60 and null boundary condition. If $2^{t-1} < n \leq 2^t$ for some positive integer t , then the group order of H is 2^t .*

3. Characteristic matrices of cellular automata

In this section, we deal with characteristic matrices of uniform CA configured with rule 60 and intermediate boundary condition. Such a matrix T of the CA is given by

$$T_{i,j} = \begin{cases} 1, & \text{if } i = j \text{ or } i = j + 1, \\ 1, & \text{if } i = 1 \text{ and } j = 3, \\ 0, & \text{otherwise} \end{cases} \quad \text{or} \quad T = \begin{pmatrix} 1 & 0 & 1 & & & & \\ 1 & 1 & 0 & & & & \\ & 1 & 1 & & & & \\ & & 1 & 1 & & & \\ & & & 1 & 1 & & \\ & & & & 1 & 1 & \\ & & & & & \ddots & \ddots \\ & & & & & & \ddots & \ddots \end{pmatrix}$$

where all the values of the blank entries are zero. From now on, all the values of the blank entries in matrix representation will always be zero unless otherwise specified.

And we can have T^2, T^3, T^4, \dots as follows;

$$T^2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot \\ & & & & & \cdot & \cdot & \cdot \end{pmatrix},$$

$$T^3 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ & & & \cdot & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$T^4 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ & & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot \\ \dots & & & & & \dots \end{pmatrix},$$

Now we will consider submatrices of T^m for a positive integer m . Let $(T^m)_L$ denote the matrix consists of the left 3 columns of T^m and $(T^m)_R$ the matrix eliminating the left 3 columns from T^m . And let $(T^m)_{LU}$ denote the matrix consists of the upper 3 rows of $(T^m)_L$ and $(T^m)_{LL}$ the matrix eliminating the upper 3 rows from $(T^m)_L$. Similarly, let $(T^m)_{RU}$ denote the matrix consists of the upper 3 rows of $(T^m)_R$ and $(T^m)_{RL}$ the matrix eliminating the upper 3 rows from $(T^m)_R$. Then T^m consists of 4 submatrices $(T^m)_{LU}$, $(T^m)_{LL}$, $(T^m)_{RU}$ and $(T^m)_{RL}$. Here we can easily know that $(T^m)_{RU}$ is always a zero matrix.

LEMMA 3.1. *Let T be the characteristic matrix of a uniform CA of length $n \geq 3$ configured with rule 60 and intermediate boundary condition. If $T^{m+l} = T^l$ for some positive integers l and m , then m is a multiple of 3.*

Proof. By the above disscussion, we have

$$T_{LU} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, (T^2)_{LU} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

$$(T^3)_{LU} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, (T^4)_{LU} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

and for each positive integer t the values of $(T^t)_{LU}$ are depend only on the values of $(T^r)_{LU}$'s where $1 \leq r \leq t$. So the sequence of $T_{LU}, (T^2)_{LU}, (T^3)_{LU}, \dots$ is the iteration of 3 matrices

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Now let $T^{m+l} = T^l$ for some positive integers l and m . Then we have $(T^{m+l})_{LU} = (T^l)_{LU}$. Thus we have the conclusion. \square

Through the proof of Lemma 3.1, we can also know that any uniform CA of length $n \geq 3$ configured with rule 60 and intermediate boundary condition could not be a group CA by Lemma 2.1.

LEMMA 3.2. *Let S be the characteristic matrix of a uniform CA of length n with $2^{t-1} < n \leq 2^t$ for a positive integer t configured with rule 60 and null boundary condition. Then, for a positive integer m , $S^{m+1} = S$ if and only if m is a multiple of 2^t .*

Proof. If m is a multiple of 2^t , then $S^m = I$ by Lemma 2.2, and so we have $S^{m+1} = S$. Now let $S^{m+1} = S$ for some positive integer m . Then $S^m = I$ since S is invertible by Lemma 2.1 and Lemma 2.2. So m is a multiple of 2^t by Lemma 2.2 again. \square

LEMMA 3.3. *Let T be the characteristic matrix of a uniform CA of length n with $3 + 2^{t-1} < n \leq 3 + 2^t$ for a positive integer t configured with rule 60 and intermediate boundary condition. If $T^{m+1} = T$ for some positive integer m , then m is a multiple of 2^t .*

Proof. Let $T^{m+1} = T$ for some positive integer m . Then $m \geq 3$ by Lemma 3.1 and we have $(T^{m+1})_{RL} = T_{RL}$ clearly. Since $(T^l)_{RU}$ is always a zero matrix for all positive integers l , the values of $(T^s)_{RL}$ for every positive integer s are depend only on the values of $(T^r)_{RL}$'s where $1 \leq r \leq s$. And there is no difference between intermediate boundary condition and null boundary condition at the rightmost cell since rule 60 applies. Now let S be the characteristic matrix of a uniform CA of length $n - 3$ configured with rule 60 and null boundary condition. Then $T_{RL} = S$, and so $(T^l)_{RL} = S^l$ for all positive integers l . Therefore we have $S^{m+1} = S$. Hence we have the conclusion by Lemma 3.2. \square

THEOREM 3.4. *Let T be the characteristic matrix of a uniform CA of length n with $3 + 2^{t-1} < n \leq 3 + 2^t$ for a positive integer t configured with rule 60 and intermediate boundary condition. If $T^{m+1} = T$ for some positive integer m , then m is a multiple of $2^t \cdot 3$.*

Proof. It is an immediate consequence of Lemma 3.1 and 3.3. \square

In fact, we can check $m = 2^t \cdot 3$ for $t = 1, 2, 3, 4$ or 5 in Theorem 3.4 by calculating $T^7 = T^{1+2^1 \cdot 3}$, $T^{13} = T^{1+2^2 \cdot 3}$, $T^{25} = T^{1+2^3 \cdot 3}$, $T^{49} = T^{1+2^4 \cdot 3}$

and $T^{97} = T^{1+2^5 \cdot 3}$, where T has a sufficiently large size, as follows;

$$T^7 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ & & & \dots & & \dots \end{pmatrix},$$

$$T^{13} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ & & & \dots & & & & \dots \end{pmatrix},$$

$$T^{25} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ & & \dots & & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & 0 & \dots \\ 1 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & \dots \\ & & \dots & & \dots & & & \dots & & \dots & \dots \end{pmatrix} \leftarrow \text{(11)th row,}$$

$$T^{49} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ & & \dots & & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & 0 & \dots \\ 1 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & \dots \\ & & \dots & & \dots & & & \dots & & \dots & \dots \end{pmatrix} \leftarrow \text{(19)th row,}$$

$$T^{97} = \left(\begin{array}{cccccccccccc} 1 & 0 & 1 & 0 & 0 & 0 & & & & & & \\ 1 & 1 & 0 & 0 & 0 & 0 & & & & & & \\ 0 & 1 & 1 & 0 & 0 & 0 & & & \dots & & & \\ 0 & 0 & 1 & 1 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & 1 & 1 & 0 & & & & & & \\ & & & \dots & & \dots & & \dots & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & 0 & \dots \\ 1 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & \dots \\ & & & \dots & & \dots & & \dots & & \dots & & \dots \end{array} \right) \leftarrow \text{(35)th row.}$$

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Department of Mathematics
 Kyungshung University
 Busan 608-736, Republic of Korea
 E-mail: jgkim@ks.ac.kr